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Thermoelastic stability of layered shallow shells

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Dedicated to Professor Bruno A. Boley

Abstract

The phenomena associated with thermal snap-through and snap-buckling of symmetrically layered shallow shells of polygonal planform are studied by means of a two-degree-of-freedom model derived from a Ritz–Galerkin approximation. The composite structure is homogenized considering perfect bond and the kinematic assumptions of the first order shear deformation theory. The simply supported shell edges are assumed to be prevented from in-plane motions. The geometrically non-linear, quasi-static equilibrium conditions are derived according to the von Kármán–Tsien theory and simplified by the Berger-approximation. A unifying non-dimensional formulation of the elastic stability analysis is presented that turns out to be independent of the special polygonal planform of the simply supported shallow shell.

Keywords: Snap-through; Snap-buckling; Polygonal planform; Non-linear modal projection; Similarity solution

1. Introduction

Temperature variations in thin-walled members of engineering structures often represent the predominant cause of failure resulting from a loss of stability, e.g., thermal buckling. Extensive studies of thermoelastic stability problems have been performed in the area of aeronautics and astronautics for slender wings heated in high-speed flight and for shells of revolution of rockets (see, e.g., Hoff, 1958). In the textbooks of Boley and Weiner (1960) and Noda et al. (2000), single chapters are devoted to thermally induced instability of thin-walled structures. The review article of Thornton (1993) compiles various phenomena of thermal buckling and post-buckling of both homogeneous and composite plates and shells. The effect of non-uniform temperature variations in geometrically non-linear plates was first considered by van der Neut (1958). Mahayni (1966) generalizes the Kármán–Tsien large-deflection equations for cylindrical shallow shells to include thermal effects and presents the solution for a mean temperature varying parabolically in the axial direction. Irschik (1986) discusses the large deflections and stability of polygonal

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thin homogeneous panels under thermal pre-stress using the approximation of Berger (1955). Comprehensive surveys on the thermoelastic stability of beams, plates, and shells are given by Ziegler and Rammerstorfer (1989) and by Tauchert (1991).

The present paper deals with the instability processes of elastic snap-through and snap-buckling of symmetrically layered shallow shells of arbitrary polygonal planform. The straight shell edges are assumed to be prevented from in-plane motions and are simply supported. The composite structure is approximately modeled by means of a first order homogenization procedure, thus rendering the boundary value problem of an effective homogeneous structure. The analysis starts with the derivation of the non-linear equilibrium conditions according to the theory of von Kármán and Tsien (1941), modified by the kinematic hypothesis of Mindlin (1951). The influence of a prescribed temperature field at sufficiently low rate is characterized by the effective mean thermal strain and the thermal curvature, respectively. The geometrical approximation of Berger (1955), that replaces the in-plane forces by a hydrostatic in-plane stress state, is adopted. Applying a multi-modal approach and the Galerkin procedure to the boundary value problem, result in a coupled cubic system of equations for the generalized coordinates. Within the two-degree-of-freedom model, elastic snap-through and snap-buckling and the narrow post-buckling ranges of stable equilibria are studied.

Using a proper non-dimensional formulation, the results turn out to be independent of the special polygonal planform of the shallow shell. Structures having the same similarity numbers exhibit an identical (non-dimensional) response. Any special, possibly non-regular, polygonal planform of the shell enters only via the eigenvalues and mode shapes of the reduced linear boundary value problem of associated simple membranes.

2. Multi-modal approach for non-linear thermoelastic deflection of shallow shells

Geometrical non-linearity is considered by means of the kinematic assumptions for the midsurface strains of the shallow shell according to von Kármán and Tsien (1941). Taking into account the effect of shear, the distribution of strain through the thickness of the shell is approximated by the Reissner–Mindlin first order shear deformation theory (see Reissner, 1985; Mindlin, 1951). Symmetrically laminated plates and shells composed of transversely isotropic layers are approximately modeled by means of a first order homogenization procedure, thus rendering the boundary value problem of an effective homogeneous structure.

The strain energy of the thermally stressed and sufficiently shallow homogenized shell can be decomposed into the membrane energy,

$$U_{\mathrm{m}} = \frac{1}{2} D \int_A [\mathrm{I}_e^2 - 2(1-\nu)\mathrm{II}_e] \, \mathrm{d}A, \tag{1}$$

the bending energy,

$$U_{\mathrm{b}} = \frac{K}{2} \int_A \left\{ \left[\psi_{x,x}^2 + \psi_{y,y}^2 + \frac{1}{2}(\psi_{x,y} + \psi_{y,x})^2 \right] + \nu \left[2\psi_{x,x}\psi_{y,y} - \frac{1}{2}(\psi_{x,y} + \psi_{y,x})^2 \right] \right\} \mathrm{d}A, \tag{2}$$

and into the parts due to shear deformation,

$$U_{\mathrm{s}} = \frac{1}{2s} \int_A \left[(w_{,x} + \psi_x)^2 + (w_{,y} + \psi_y)^2 \right] \mathrm{d}A, \tag{3}$$

and the thermoelastic energy,

$$U_{\mathrm{th}} = -(1+\nu) \left[K \int_A \kappa_\theta (\psi_{x,x} + \psi_{y,y}) \, \mathrm{d}A + D \int_A n_\theta \mathrm{I}_e \, \mathrm{d}A \right], \tag{4}$$

$$\mathrm{I}_e = e_{xx} + e_{yy}, \quad \mathrm{II}_e = e_{xx}e_{yy} - \frac{1}{4}e_{xy}^2, \tag{5}$$

represent the first and the second invariant of the midsurface strain tensor, respectively, where e_{ij}, $i, j = x, y$, stand for the components of the midsurface strains. According to Nash and Modeer (1960), who have generalized the assumption of Berger (1955) in pre-stressed plate theory, by including the curvature of shallow shells, the second invariant II_e is neglected in Eq. (1), thus the membrane energy is approximately given by the midsurface integral

$$U_\mathrm{m} \cong \frac{1}{2}D\int_A \mathrm{I}_e^2\,\mathrm{d}A. \tag{6}$$

This is a reasonable well-behaving approximation for structures with immovable boundary conditions. In Eqs. (1)–(4), w denotes the deflection and ψ_x, ψ_y are the cross-sectional rotations. A denotes the shell area projected onto the (x, y)-plane, D, K, $1/s$ represent the effective in-plane-, bending-, and shear-stiffness, respectively, resulting from the homogenization procedure of symmetrically laminated shells composed of transversely isotropic layers, $z = 0$ refers to the midsurface of the shell,

$$D = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\frac{E_k}{(1-v_k^2)}\,\mathrm{d}z, \quad K = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\frac{E_k}{(1-v_k^2)}z^2\,\mathrm{d}z, \quad \frac{1}{s} = \kappa^2\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}G_{ck}\,\mathrm{d}z, \quad v = \frac{1}{K}\sum_{k=1}^{N}K_k v_k. \tag{7}$$

v is an effective Poisson's ratio, κ^2 stands for a shear factor, and E_k and G_{ck} denote the modulus of elasticity and the transverse shear modulus in the kth layer, respectively.

The influence of thermal heating at sufficiently low rate is characterized by the effective mean thermal strain n_θ and the thermal curvature κ_θ, respectively,

$$n_\theta = \frac{1}{D(1+v)}\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\frac{E_k\alpha_k\theta}{(1-v_k)}\,\mathrm{d}z, \quad \kappa_\theta = \frac{1}{K(1+v)}\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\frac{E_k}{(1-v_k)}\alpha_k\theta z\,\mathrm{d}z, \tag{8}$$

where $\theta(x, y; z)$ denotes the prescribed temperature field, and α_k is the linear coefficient of thermal expansion in the kth layer.

The first variation of the strain energy must vanish,

$$\delta(U_\mathrm{m} + U_\mathrm{b} + U_\mathrm{s} + U_\mathrm{th}) = 0, \tag{9}$$

and thus yields the equilibrium conditions,

$$D[\mathrm{I}_e - (1+v)n_\theta]_{,i} = 0, \quad i = x, y, \tag{10}$$

$$-K\left[(\psi_{x,x} + v\psi_{y,y})_{,x} + \frac{(1-v)}{2}(\psi_{x,y} + \psi_{y,x})_{,y} - (1+v)\kappa_{\theta,x}\right] + \frac{1}{s}(w_{,x} + \psi_x) = 0, \tag{11}$$

$$-K\left[(\psi_{y,y} + v\psi_{x,x})_{,y} + \frac{(1-v)}{2}(\psi_{x,y} + \psi_{y,x})_{,x} - (1+v)\kappa_{\theta,y}\right] + \frac{1}{s}(w_{,y} + \psi_y) = 0, \tag{12}$$

$$-D[\mathrm{I}_e - (1+v)n_\theta](\Delta w - 2H) - \frac{1}{s}(\Delta w + \psi_{x,x} + \psi_{y,y}) = 0, \tag{13}$$

where

$$H = (k_x + k_y)/2 \tag{14}$$

is the mean curvature of the middle surface. After eliminating the cross-sectional rotations, a single fourth-order differential equation for the shell midsurface deflection $w(x,y)$ is obtained:

$$\mathsf{D}\{w\} = -(1+\nu)K\,\Delta\kappa_\theta, \tag{15}$$

where the operator

$$\mathsf{D}\{w\} = K(1+sn)\,\Delta\Delta w - n[\Delta w - 2(H - Ks\,\Delta H)] \tag{16}$$

characterizes the pre-stressed shallow shell, and n denotes an isotropic in-plane force, that is constant throughout the midsurface. The latter, in the Berger-approximation, is related to the deflection by the averaging integral (see, e.g., Heuer, 1994),

$$n = D[\mathrm{I}_e - (1+\nu)n_\theta] = \text{const.} = -D\left[\frac{1}{2A}\int_A w(\Delta w - 4H)\,\mathrm{d}A + \bar{n}_\theta\right], \tag{17}$$

where the influence of heating is taken care of by the averaged thermal strain

$$\bar{n}_\theta = \frac{(1+\nu)}{A}\int_A n_\theta\,\mathrm{d}A. \tag{18}$$

Shear-deformable plates and shells with straight edges and boundary conditions of the hard hinged type are considered subsequently, thus the boundary conditions are

$$\Gamma : w = 0, \quad m_n = 0, \quad \psi_s = 0, \tag{19}$$

which in the case of straight edges become the classical form of thermoelasticity of simply supported plates,

$$\Gamma : w = 0, \quad \Delta w = -(1+\nu)\kappa_\theta. \tag{20}$$

In a multi-modal approach the deflection and the thermal curvature of the shallow shell are expanded into the orthogonal set of eigenfunctions $w_j^*(\mathbf{x})$,

$$w(\mathbf{x}) = \sum_{j=1}^{N} c_j w_j^*(\mathbf{x}), \quad \kappa_\theta(\mathbf{x}) = \sum_{j=1}^{N} c_{\theta j} w_j^*(\mathbf{x}), \quad c_{\theta j} = \int_A \kappa_\theta(\mathbf{x}) w_j^*(\mathbf{x})\,\mathrm{d}A \bigg/ \int_A w_j^{*2}(\mathbf{x})\,\mathrm{d}A, \tag{21}$$

where w_j^* are the solutions of the homogeneous linearized problem of a flat plate that is equally shaped as the shell's base plane. The corresponding boundary value problem is given by the reduced Eqs. (15) and (20), i.e.,

$$A : K(1 - sD\bar{n}_{j\theta}^{\mathrm{P}})\,\Delta\Delta w_j^* + D\bar{n}_{j\theta}^{\mathrm{P}}\,\Delta w_j^* = 0, \tag{22}$$

$$\Gamma : w = 0, \quad \Delta w = 0 \tag{23}$$

with the mean thermal strain of the flat plate critical in the jth mode, $\bar{n}_{j\theta}^{\mathrm{P}}$. In Eq. (21), the amplitude coefficients c_j, $c_{\theta j}$ carry the appropriate dimensions, and the superscript $(*)$ stands for non-dimensional quantities. Irschik (1985) has shown that the eigenfunctions of those shear-deformable simply supported plates with polygonal planform are governed by a set of second-order Helmholtz-differential equations with homogeneous Dirichlet-boundary conditions,

$$\Delta w_j^* + \alpha_j w_j^* = 0, \quad j = 1, 2, \ldots, N, \tag{24}$$

$$\Gamma : w_j^* = 0, \tag{25}$$

α_j is the jth eigenvalue of an effectively pre-stressed membrane of the same shape as the plate and it is related to the jth critical mean thermal strain through

$$\bar{n}^{\mathrm{P}}_{j\theta} = K\alpha_j / D(1 + Ks\alpha_j). \tag{26}$$

Using Eq. (21) as a Ritz-approximation for the solution of Eq. (15) and applying Galerkin's procedure (see, e.g., Ziegler, 1998, p. 593 ff.),

$$\int_A [\mathrm{D}\{w\} + (1+\nu)K\,\Delta\kappa_\theta] c_k w_k^*\,\mathrm{d}A = 0 \tag{27}$$

yield a coupled system of algebraic cubic equations for the unknowns c_j:

$$\sum_{k=1}^{N} a_{jk}c_k + c_j \sum_{k=1}^{N} e_{jk}c_k^2 + \sum_{k=1}^{N} b_{jk}c_k^2 + c_j \sum_{k=1}^{N} d_{jk}c_k = f_{\theta j}, \quad j = 1, 2, \ldots, N, \tag{28}$$

with the coefficients

$$\left.\begin{aligned} a_{jk} &= \left\{ \left[K\alpha_j^2 - \bar{n}_\theta D\alpha_j(1+Ks\alpha_j)\right]\delta_{jk} + \tfrac{4D\vartheta_k^*}{L_0^2\beta_j^*}(\vartheta_j^* - Ks\delta_j^*/L_0^2) \right\} \Big/ (1+Ks\alpha_j), \\ b_{jk} &= \tfrac{D\alpha_k\beta_k^*}{L_0\beta_j^*}(\vartheta_j^* - Ks\delta_j^*/L_0^2)/(1+Ks\alpha_j), \quad d_{jk} = \tfrac{2D\alpha_j\vartheta_k^*}{L_0}, \quad e_{jk} = \tfrac{D\alpha_j\alpha_k\beta_k^*}{2}, \\ f_{\theta j} &= \left[\tfrac{2\bar{n}_\theta D}{L_0\beta_j^*}(\vartheta_j^* - Ks\delta_j^*/R_0^2) + c_{\theta j}(1+\nu)K\alpha_j \right] \Big/ (1+Ks\alpha_j). \end{aligned}\right\} \tag{29}$$

L_0 stands for a characteristic length of the shell structure, δ_{jk} is the Kronecker delta, and the non-dimensional quantities

$$\beta_j^* = \frac{1}{A}\int_A w_j^{*2}\,\mathrm{d}A, \quad \vartheta_j^* = \frac{L_0}{A}\int_A Hw_j^*\,\mathrm{d}A, \quad \delta_j^* = \frac{L_0^3}{A}\int_A \Delta Hw_j^*\,\mathrm{d}A \tag{30}$$

characterize the norm of w_j^* and the influence of the mean curvature H, respectively.

Subsequently it is assumed that the initial curvature of the shallow shell is proportional to the basic eigenmode of the linearized plate problem w_1^*. Consequently, Eq. (14) becomes

$$H = \frac{1}{2}(k_x + k_y) = -\frac{1}{2}C_0\,\Delta w_1^* = \frac{1}{2}C_0\alpha_1 w_1^*, \tag{31}$$

and, with $j = 1$ in Eq. (30),

$$\beta_1^* = \frac{1}{A}\int_A w_1^{*2}\,\mathrm{d}A, \quad \vartheta_1^* = \frac{1}{2}C_0L_0\alpha_1\beta_1^*, \quad \delta_1^* = -\frac{1}{2}C_0L_0^3\alpha_1^2\beta_1^*, \quad \vartheta_j^* = \delta_j^* = 0, \ldots, j \geqslant 2. \tag{32}$$

This special case allows the evaluation of the first and fourth summation in Eq. (28) which, in its non-dimensional form, simplifies to

$$\left[a_j^*c_j^* + c_j^*\sum_{k=1}^{N} e_{jk}^*c_k^{*2}\right] + \delta_{1j}\sum_{k=1}^{N} b_{jk}^*c_k^{*2} + d_{j1}^*c_j^*c_1^* = f_j^*c_{\theta j}^* + f_{n_\theta}^*\bar{n}_\theta\delta_{1j}, \quad j = 1, 2, \ldots, N, \tag{33}$$

where

$$\left.\begin{aligned} a_j^* &= 1 - \tfrac{\bar{n}_\theta}{\bar{n}^{\mathrm{P}}_{j\theta}} + C_0^{*2}D^*\beta_1^*\tfrac{(1+s^*\alpha_j^*)}{\alpha_j^{*2}}\delta_{1j}, \quad b_{1j}^* = \tfrac{1}{2}C_0^*D^*\beta_j^*\tfrac{(1+s^*\alpha_j^*)}{\alpha_j^*}, \\ d_{j1}^* &= 2b_{1j}^*\tfrac{\beta_1^*}{\beta_j^*}, \quad e_{jk}^* = \tfrac{1}{2}D^*\alpha_k^*\beta_k^*\tfrac{(1+s^*\alpha_j^*)}{\alpha_j^*}, \quad f_{n_\theta}^* = \tfrac{C_0^*D^*(1+s^*)}{L_0^2\alpha_1}, \quad f_j^* = \tfrac{(1+\nu)}{L_0^2\alpha_1\alpha_j^{*2}}, \\ \bar{n}^{\mathrm{P}}_{j\theta} &= \tfrac{\alpha_j^*\alpha_1L_0^2}{D^*(1+\alpha_j^*s^*)}, \quad c_j^* = \tfrac{c_j}{L_0}, \quad c_{\theta j}^* = c_{\theta j}L_0. \end{aligned}\right\} \tag{34}$$

The non-dimensional similarity numbers are, for β_j^* see Eq. (30),

$$\alpha_j^* = \frac{\alpha_j}{\alpha_1}, \quad s^* = Ks\alpha_1, \quad D^* = \frac{DL_0^2}{K}, \quad C_0^* = \frac{C_0}{L_0}. \tag{35}$$

Structures having the same similarity numbers exhibit an identical non-dimensional response. Any special, possibly non-regular, polygonal planform of the shell enters only via the parameters α_j^* and β_j^* of the reduced linear boundary value problem defined by Eqs. (24) and (25).

For the case of a flat plate, i.e., when $H = C_0 = 0$, the left side of Eq. (33) reduces to the first two terms (indicated in brackets) and consequently the right side to the first term only,

$$a_j^* c_j^* + c_j^* \sum_{k=1}^{N} e_{jk}^* c_k^{*2} = f_j^* c_{\theta j}^*, \quad j = 1, 2, \ldots, N. \tag{36}$$

3. Thermal stability

It is well-known from the stability analysis of shallow arches (see, e.g., Ashwell, 1962) that at least a two-mode approximation must be considered to find the critical loads of lower order, and thus to account for the expected processes of snap-through and snap-buckling.

Consequently Eq. (33) in such a two-mode approximation gives the pair of coupled cubic equations

$$a_1^* c_1^* + e_{11}^* c_1^{*3} + e_{12}^* c_1^* c_2^{*2} + 3b_{11}^* c_1^{*2} + b_{12}^* c_2^{*2} = f_1^* c_{\theta 1}^* + f_{n_\theta}^* \bar{n}_\theta, \tag{37}$$

$$a_2^* c_2^* + e_{21}^* c_1^{*2} c_2^* + e_{22}^* c_2^{*3} + d_{21}^* c_1^* c_2^* = 0. \tag{38}$$

Reduction of the system of Eqs. (37) and (38) is achieved by dividing Eq. (38) by the generalized coordinate c_2^* and subsequently by transforming the variables

$$\bar{c}_1^* = c_1^* + \frac{b_{11}^*}{e_{11}^*} \equiv c_1^* + C_0^*, \tag{39}$$

and

$$\bar{c}_2^* = c_2^* \sqrt{e_{12}^*/e_{11}^*} \equiv c_2^* \sqrt{\alpha_2^* \beta_2^* / \beta_1^*}. \tag{40}$$

Substitution yields,

$$\bar{a}_1^* \bar{c}_1^* + \bar{c}_1^{*3} + \bar{c}_1^* \bar{c}_2^{*2} - \bar{f}_1^* = 0, \tag{41}$$

$$\bar{a}_2^* + \bar{c}_1^{*2} + \bar{c}_2^{*2} = 0, \tag{42}$$

where the remaining coefficients become

$$\left.\begin{aligned}
\bar{a}_1^* &= \tfrac{a_1^*}{e_{11}^*} - 3\tfrac{b_{11}^{*2}}{e_{11}^{*2}} = \tfrac{a_1^{*\mathrm{P}}}{e_{11}^*} - C_0^{*2}, \quad a_1^{*\mathrm{P}} = 1 - \tfrac{\bar{n}_\theta}{\bar{n}_{1\theta}^{\mathrm{P}}}, \\
\bar{a}_2^* &= \tfrac{a_2^*}{e_{21}^*} - \tfrac{d_{21}^* b_{11}^*}{e_{21}^* e_{11}^*} + \tfrac{b_{11}^{*2}}{e_{11}^{*2}} = \tfrac{a_2^{\mathrm{P}*}}{e_{21}^*} - C_0^{*2}, \quad a_2^{\mathrm{P}*} = 1 - \tfrac{\bar{n}_\theta}{\bar{n}_{2\theta}^{\mathrm{P}}}, \\
e_{11}^* &= \tfrac{1}{2} D^* \beta_1^* (1 + s^*), \quad e_{21}^* = \tfrac{1}{2} D^* \beta_1^* \tfrac{(1+s^*\alpha_2^*)}{\alpha_2^*}, \\
\bar{f}_1^* &= \tfrac{(f_1^* c_{\theta 1}^* + f_{n_\theta}^* \bar{n}_\theta)}{e_{11}^*} + a_1^* \tfrac{b_{11}^*}{e_{11}^*} - 2\tfrac{b_{11}^{*3}}{e_{11}^{*2}} = \tfrac{1}{e_{11}^*}\left[C_0^* + \tfrac{(1+\nu)}{L_0^2 \alpha_1} c_{\theta 1}^*\right].
\end{aligned}\right\} \tag{43}$$

Subsequently the two cases of instability inherent in Eqs. (41) and (42) are studied separately together with their post-buckling ranges.

3.1. Single-mode approximation: thermal snap-through

Snap-through behavior of shallow shells is considered using the classical single-mode approximation, i.e., the post-critical deformation is assumed to be affine to the first mode. Consequently, Eq. (41) remains as the characteristic equation of equilibrium by putting $\bar{c}_2^* = 0$,

$$\bar{c}_1^{*3} + \bar{a}_1^* \bar{c}_1^* - \bar{f}_1^* = 0. \tag{44}$$

Eq. (44) reflects a flat plate single-mode approximation, i.e., $j = 1$, however, the effective coefficients $\bar{a}_1^*$, $\bar{f}_1^*$ given in Eq. (43), and the transformed generalized coordinate $\bar{c}_1^*$ expressed by Eq. (39) are to be noted. Stability requires a positive second derivative of the potential energy and thus, the stability limit is derived from the first derivative of Eq. (44), the equilibrium condition,

$$3\bar{c}_1^{*2} + \bar{a}_1^* = 0. \tag{45}$$

Real solutions require $\bar{a}_1^* \leqslant 0$ in Eq. (45), see again the first coefficient in Eq. (43). The critical position where snap-through may be initiated is determined by solving the pair of Eqs. (44) and (45), rendering a critical value of the generalized coordinate in Eq. (39),

$$\bar{c}_1^* = (\bar{c}_1^*)_{c1} = \frac{3}{2}\frac{(\bar{f}_1^*)_{c1}}{\bar{a}_1^*}, \quad (\bar{c}_1^*)_{c1} < 0. \tag{46}$$

Fig. 1 shows the equilibrium chart in the $(\bar{f}_1^*, \bar{c}_1^*)$-plane for various parameters $\bar{a}_1^*$, illustrating the solutions of Eq. (44), and indicating that snap-through can occur only for $\bar{a}_1^* \leqslant 0$. Substituting the critical value of Eq. (46) into the limiting condition of stability, Eq. (45), yields what is called a cusp catastrophe in the parameter plane (see, e.g., Troger and Steindl, 1991),

$$\frac{4}{27}\bar{a}_1^{*3} + (\bar{f}_1^{*2})_{c1} = 0 \Rightarrow (\bar{f}_1^{*2})_{c1} = \frac{4}{27}|\bar{a}_1^*|^3. \tag{47}$$

In the post-buckling range, the inequality holds which can be expressed by

$$\bar{f}_1^{*2} > \frac{4}{27}|\bar{a}_1^*|^3. \tag{48}$$

Putting the shell curvature to zero, i.e., $C_0^* = 0$, in Eq. (43), yields in the limiting process the well-known flat plate parameters of the homogenized structure as,

$$\lim_{C_0^* \to 0} \bar{a}_1^* = \frac{a_1^{*\mathrm{P}}}{e_{11}^*} \equiv \frac{1}{e_{11}^*}\left(1 - \frac{\bar{n}_\theta}{\bar{n}_{1\theta}^{\mathrm{P}}}\right), \quad \lim_{C_0^* \to 0} \bar{f}_1^* = \frac{1}{e_{11}^*}\frac{(1+\nu)}{L_0^2 \alpha_1} c_{\theta 1}^*. \tag{49}$$

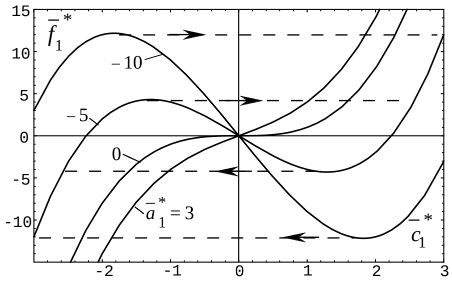

Fig. 1. Snap-through: equilibrium chart in the $(\bar{f}_1^*, \bar{c}_1^*)$-plane for various parameters $\bar{a}_1^*$ according to Eq. (44).

The coefficient e^*_{11}, also given in Eq. (43), remains unchanged and is to be substituted together with Eq. (49) in Eqs. (44)–(48).

3.2. Two-mode approximation: thermal snap-buckling

As was pointed out by Ashwell (1962) (see also Irschik, 1986; Ziegler and Rammerstorfer, 1989) the loss of stability of the static equilibrium position may be accompanied by a bifurcation to a higher mode. For non-linear random vibrations of buckled plates, this so-called snap-buckling has been studied by Heuer et al. (1993). Consequently, the full system of the coupled cubic equations (41) and (42) must be considered. Its solution which could be made explicit in the transformed domain, defines the range of snap-buckling,

$$\bar{c}^*_1 = \frac{\bar{f}^*_1}{(\bar{a}^*_1 - \bar{a}^*_2)}, \quad \bar{c}^{*2}_2 = -(\bar{c}^{*2}_1 + \bar{a}^*_2) \equiv -\left(\frac{\bar{f}^{*2}_1}{(\bar{a}^*_1 - \bar{a}^*_2)^2} + \bar{a}^*_2\right). \tag{50}$$

The first equation of (50) shows proportionality between $\bar{c}^*_1$ and $\bar{f}^*_1$, the second one defines an ellipse that corresponds to real solutions in the case of $\bar{c}^{*2}_2 > 0$. The critical position where snap-buckling may be initiated is determined from the condition

$$\bar{c}^{*2}_2 = 0 \Rightarrow \bar{c}^{*2}_1 + \bar{a}^*_2 = 0, \tag{51}$$

which, when substituted into the first of Eq. (50), yields the corresponding critical load of bifurcation

$$(\bar{f}^{*2}_1)_{c2} = |\bar{a}^*_2|(\bar{a}^*_1 - \bar{a}^*_2)^2. \tag{52}$$

Fig. 2 illustrates the $(\bar{f}^*_1, \bar{c}^*_i)$-paths, $i = 1, 2$, according to Eq. (50), exhibiting the typical bifurcation behavior of snap-buckling.

However, this type of two-mode bifurcation occurs only if $(\bar{f}^*_1)_{c2} < (\bar{f}^{*2}_1)_{c1}$, that condition corresponds to the range of the parameters $\bar{a}^*_1$ and $\bar{a}^*_2$, derived by considering Eqs. (47) and (52),

$$\frac{3}{4}|\bar{a}^*_2| \leqslant |\bar{a}^*_1| \leqslant 3|\bar{a}^*_2|. \tag{53}$$

Finally, when considering both critical states, one of snap-through and of snap-buckling, the range of equilibrium is formally defined by

$$\bar{f}^{*2}_1 \leqslant |\bar{a}^*_2|(\bar{a}^*_1 - \bar{a}^*_2)^2 \leqslant \frac{4}{27}|\bar{a}^*_1|^3. \tag{54}$$

Fig. 3 shows the equilibrium chart in the $(\bar{a}^*_1, \bar{f}^*_1)$-plane according to Eq. (54) for a chosen parameter of $\bar{a}^*_2 = -2$ where the shaded domain indicates the stable range of limited validity due to the two-mode approximation.

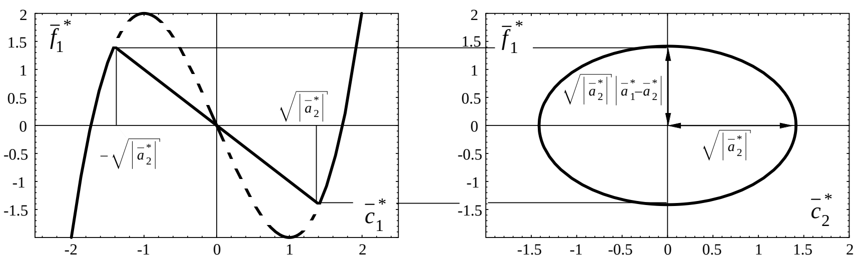

Fig. 2. Snap-buckling: $(\bar{f}^*_1, \bar{c}^*_i)$-paths, $i = 1, 2$, according to Eq. (50).

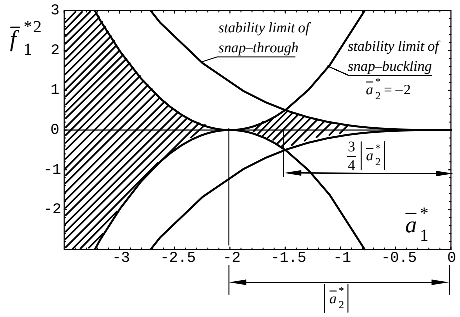

Fig. 3. Equilibrium chart in the $(\bar{a}_1^*, \bar{f}_1^*)$-plane according to Eq. (54) for a chosen parameter of $\bar{a}_2^* = -2$; the shaded domain indicates the stable range of limited validity due to the two-mode approximation.

The parameters of the flat plate problem are found again by putting the shell curvature to zero, i.e., $C_0^* = 0$, rendering

$$\lim_{C_0^* \to 0} \bar{a}_2^* = \frac{a_2^{*P}}{e_{21}^*} \equiv \frac{1}{e_{21}^*}\left(1 - \frac{\bar{n}_\theta}{\bar{n}_{2\theta}^{P}}\right), \tag{55}$$

where e_{21}^* given in Eq. (43) remains unchanged, and $\bar{a}_1^*$, $\bar{f}_1^*$ are determined from Eq. (49).

4. Conclusions

Symmetrically layered shallow shells are homogenized and the resulting first order elastic stability limits of the resulting structure are derived. A two-degree-of-freedom model derived from a Ritz–Galerkin approximation serves sufficiently well to analyze snap-through and snap-buckling and the narrow post-buckling range of stable equilibria. Similarity law and a non-dimensional analysis make the results, cast into the parameter space, readily available with especially simple solutions for simply supported (immovable) straight edges of a polygonal planform. Numerically the eigenvalues and mode shapes of associated simple membranes must be determined, e.g., by applying the boundary element method.

Results for thermally loaded plates are included by putting the initial curvature of the shell to zero.

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